Polynomial Cocycles of Alexander Quandles and Applications

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Abstract

Cocycles are constructed by polynomial expressions for Alexander quandles. As applications, non-triviality of some quandle homology groups are proved, and quandle cocycle invariants of knots are studied. In particular, for an infinite family of quandles, the non-triviality of quandle homology groups is proved for all odd dimensions.

1 Introduction

Quandles are self-distributive sets with additional properties (see below for details). They have been used in the study of knots since 1980s. Cohomology theories of quandles were developed as modifications of rack cohomology theory [10], and their cocycles have been used to construct invariants of knots and knotted surfaces [7]. Quandle cohomology theory was further generalized [5, 6, 10].

To compute the quandle cocycle invariant, explicit cocycles are required. At first the computations relied on cocycles found by computer calculations. A significant progress was made in computations of the invariants after Mochizuki discovered a family of 2- and 3-cocycles for dihedral and other linear Alexander quandles, for which cocycles were written by polynomial expressions. Formulas for important families of knots and knotted surfaces and their applications followed [3, 12, 13]. Homology groups for higher dimensions are studied in [18] for dihedral quandles.

In this paper, following the method of the construction by Mochizuki, a variety of n-cocycles for $n \geq 2$ are constructed for some Alexander quandles by polynomial expressions. As applications, we prove the non-triviality of quandle homology groups for an infinite family of Alexander quandles for all odd dimensions. These cocycles are also used to compute the invariants for some families of knots. Much of the results are based on the Ph.D. dissertation by K.A. [1].

The paper is organized as follows. In Section 2, a brief review of quandle homology groups are given and a key lemma is proved on polynomial cocycles. Quandle cocycle invariants of knots are studied in Section 3, and non-triviality of quandle homology groups is proved in Section 4.

2 Preliminaries

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ satisfying the three conditions: (I) For any $a \in X$, $a * a = a$, (II) for any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$, and (III)

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for any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$. A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [4, 9, 14, 15]. For $\Lambda = \mathbb{Z}[t, t^{-1}]$, any Λ-module M is a quandle with $a * b = ta + (1-t)b$, $a, b \in M$, that is called an *Alexander quandle*.

A cohomology theory of quandles was defined [7] as a modification of rack cohomology theory [10] as follows. Let $C_n^R(X)$ be the free abelian group generated by n-tuples $(x_1, \ldots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^R(X) \longrightarrow C_{n-1}^R(X)$ by

$$\begin{aligned}\partial_n(x_1, \ldots, x_n) &= \sum_{i=2}^{n} (-1)^i [(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &- (x_1 * x_i, x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)]\end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*^R(X) = \{C_n^R(X), \partial_n\}$ is a chain complex. Let $C_n^D(X)$ be the subset of $C_n^R(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$ if $n \geq 2$; otherwise let $C_n^D = 0$. If X is a quandle, then $\partial_n(C_n^D) \subset \partial_{n-1}(C_n^D)$ and $C_*^D(X) = \{C_n^D(X), \partial_n\}$ is a subcomplex of $C_*^R(X)$. Put $C_n^Q = C_n^R(X)/C_n^D(X)$ and $C_*^Q(X) = \{C_n^Q(X), \partial_n'\}$ where, ∂_n' is the induced homomorphism. Henceforth, all boundary maps may be denoted by ∂_n. The superscripts R, Q and D, respectively, represent rack, quandle, and degenerate chain complexes. For an Abelian group G, define the chain and the cochain complexes by

$$\begin{aligned} C_*^W(X; G) &= C_*^W(X) \otimes G \qquad \partial = \partial \otimes \mathrm{id}, \\ C_W^*(X; G) &= \mathrm{Hom}(C_*^W(X), G) \qquad \delta = \mathrm{Hom}(\partial, id) \end{aligned}$$

in the usual way, where $W = D, R, Q$. The groups of cycles and boundaries are denoted respectively by $ker(\partial) = Z_n^W(X; G) \subset C_n^W(X; G)$ and $\mathrm{Im}(\partial) = B_W^n(X; G) \subset C_W^n(X; G)$ while the cocycles and coboundaries are denoted respectively by $Z_W^n(X; G)$ and $B_W^n(X; G)$, respectively. The nth quandle homology group with coefficient group G is defined by

$$H_n^Q(X; G) = H_n(C_*^w(X; G)) = Z_n^Q(X; G)/B_n^Q(X; G),$$

and the quandle cohomology group with coefficient group G, $H_Q^n(X; G)$, is defined similarly. The following is the Key Lemma of the paper.

Lemma 2.1 *Let $X = \mathbb{Z}_p[t, t^{-1}]/(g(t))$ for some prime p. Let $a_i = p^{m_i}$, for $i = 1, \ldots, n-1$, where m_i are non-negative integers. For a positive integer n, let $f : X^n \to X$ be defined by*

$$f(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^{a_1} (x_2 - x_3)^{a_2} \cdots (x_{n-1} - x_n)^{a_{n-1}} x_n^{a_n}.$$

1. *If $a_n = 0$, then f is an n-cocycle ($\in Z_{\mathrm{Q}}^n(X; X)$).*
2. *If $a_n = p^{m_n}$ for a positive integer m_n, then f is an n-cocycle if $g(t)$ divides $1 - t^a$, where $a = a_1 + a_2 + \cdots + a_{n-1} + a_n$.*

Proof. From the definition of δ, for $i = 1, \ldots, n$, we compute using the notation $y_i = x_i - x_{i+1}$

$$\begin{aligned} &\delta f(x_1, \ldots, x_{n+1}) \\ &= \sum_{i=2}^{n+1} (-1)^i [f(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) - f(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_{n+1})] \end{aligned}$$

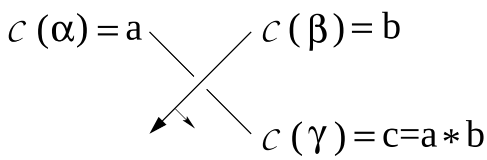

Figure 1: Quandle relation at a crossing

$$\begin{aligned}
&= \sum_{i=2}^{n}(-1)^i y_1^{a_1} y_2^{a_2} \cdots (y_{i-1}+y_i)^{a_{i-1}} y_{i+1}^{a_i} \cdots y_n^{a_{n-1}} x_{n+1}^{a_n} + (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} \\
&- \sum_{i=2}^{n}(-1)^i (ty_1)^{a_1} (ty_2)^{a_2} \cdots (ty_{i-1}+y_i)^{a_{i-1}} \cdots y_n^{a_{n-1}} x_{n+1}^{a} \\
&- (-1)^{n+1} t^a y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&(-1)^n y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n} - (-1)^n t^{a_1+\cdots+a_{n-1}} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a_n} \\
&+(-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} - (-1)^{n+1} t^{a_1+\cdots a_{n-1}} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} (ty_n + x_{n+1})^{a_n}.
\end{aligned}$$

If $a_n = 0$ then we see $\delta f(x_1, .., x_{n+1}) = 0$. If $a_n = p^{m_n}$ then we have

$$\begin{aligned}
&\delta f(x_1, \ldots, x_{n+1}) \\
&\quad = (-1)^n y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} x_{n+1}^{a} + (-1)^{n+1} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}} x_n^{a_n} - (-1)^{n+1} t^{a_1+\cdots+a_{n-1}+a_n} y_1^{a_1} \cdots y_n^{a_n} \\
&\quad = (-1)^{n+1}(1-t^a) y_1^{a_1} \cdots y_n^{a_n} \quad = \quad 0 \in X
\end{aligned}$$

by assumption. Hence f is an n-cocycle. □

For the purpose of constructing knot invariants for applications in later sections, we mainly use quandle 2- and 3-cocycles. The properties of these cocycles that we need for the knot invariants are specifically formulated as follows. A quandle 2-cocycle ϕ is regarded as a function $\phi : X \times X \to A$ with the 2-cocycle condition

$$\phi(x, y) + \phi(x * y, z) = \phi(x, z) + \phi(x * z, y * z)$$

for all $x, y, z \in X$, and $\phi(x, x) = 0$ for all $x \in X$. A quandle 3-cocycle θ is regarded as a function $\theta : X \times X \times X \to A$ with the 3-cocycle condition

$$\theta(x, z, w) + \theta(x, y, z) + \theta(x * z, y * z, w) = \theta(x * z, y, w) + \theta(x, y, w) + \theta(x * w, y * w, z * w)$$

for any $x, y, z, w \in X$, and $\theta(x, x, y) = 0$, $\theta(x, y, y) = 0$ for all $x, y \in X$.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane, see Fig. 1. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 1 holds. The (ordered) colors $\mathcal{C}(\alpha)$, $\mathcal{C}(\beta)$ are called *source* colors. Let $\mathrm{Col}_X(K)$ denote the set of colorings of a knot diagram K by a quandle X.

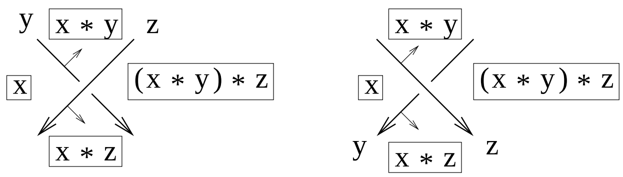

Figure 2: Quandle colorings of regions

For a coloring $\mathcal{C}$, there is a coloring of regions that extend $\mathcal{C}$ as depicted in Fig. 2. Let $(x, y, z) = (x_\tau, y_\tau, z_\tau)$ be the colors near a crossing τ such that x is the color of the region (called the source region) from which both orientation normals of over- and under-arcs point, y is the color of the under-arc (called the source under-arc) from which the normal of the over-arc points, and z is the color of the over-arc. See Fig. 2.

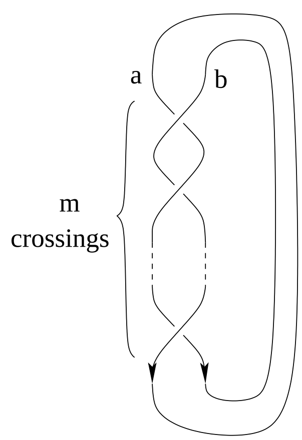

Figure 3: Colorings of torus knots

Example 2.2 Let X be an Alexander quandle. Let $T(2, m)$ be the $(2, m)$-torus knot that is the closure of the closed braid of 2-string braid with m positive crossings for a positive integer m. For a negative integer m, $T(2, m)$ consists of $|m|$ negative crossings. Denote by ξ_k the polynomial

$$\xi_k = \xi_k(t) = \sum_{i=0}^{k-1} (-t)^i,$$

and define $\xi_0 = 0$ as convention. Note that for $m > 1$, the polynomial $\xi_m(t)$ is the Alexander polynomial $\Delta_{T(2,m)}(t)$ of the knot $T(2, m)$ (see, for example, [17]).

Then it is seen by induction that if $(a, b) \in X \times X$ is the top color vector (the elements a and b are assigned to the top left and right arcs of a 2-string braid, respectively), of a coloring of $T(2, m)$ by X, then the kth color vector (the pair of colors after kth crossing) is $(a + \xi_k(b - a), b + \xi_k(a - b))$ where $1 \le k \le m$. In particular, any top color vector extends to a coloring of $T(2, m)$ for the quandle $X = \mathbb{Z}_p[t, t^{-1}]/(\xi_m(t))$.

Another example, a coloring of twist knots (see for example [19]) is depicted in Fig. 4. We denote the twist knot with with $2n + 2$ crossings as depicted in the figure by $k(2n)$.

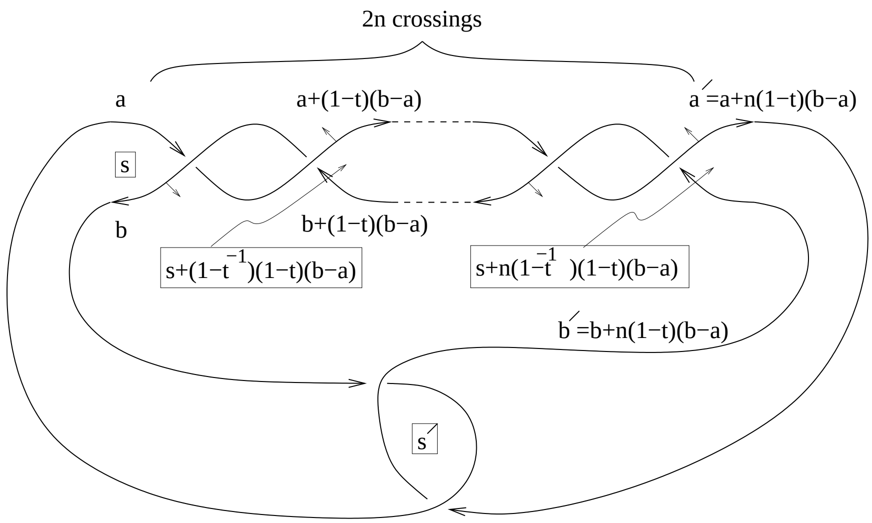

Figure 4: Colorings of twist knots

The cocycle invariant for classical knots [7] was defined as follows. Let $\phi \in Z^2_{\rm Q}(X;A)$ be a quandle 2-cocycle of a finite quandle X with the coefficient group A. Let $\mathcal{C}$ be a coloring of a given knot diagram K by X. The *Boltzmann weight* $B(\mathcal{C},\tau)$ at a crossing τ of K is then defined by $B(\mathcal{C},\tau)=\phi(x_\tau,y_\tau)^{\epsilon(\tau)}$, where x_τ, y_τ are source colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ. In Fig. 1, it is a positive crossing if the under-arc is oriented downward. Here $B(\mathcal{C},\tau)$ is an element of A written multiplicatively. The formal sum (called a state-sum) in the group ring $\mathbb{Z}[A]$ defined by $\Phi_\phi(K)=\sum_{\mathcal{C}\in {\rm Col}_{\rm X}({\rm K})}\prod_\tau B(\mathcal{C},\tau)$ is called the *quandle cocycle invariant*. The invariant is also defined by $\{\sum_\tau \epsilon(\tau)\phi(x_\tau,y_\tau) \mid \mathcal{C}\in {\rm Col}_{\rm X}({\rm K})\}$ as a multiset, in which case the values of the cocycles are written additively. If the quandle X is finite, the invariant as a multiset can be written by an expression similar to those for the state-sums as follows. Suppose a multiset of group elements is given by $\{\sqcup_{m_1} g_1,\ldots,\sqcup_{m_\ell} g_\ell\}$, where $\sqcup_{m_i} g_i$ denotes m_i copies of g_i (the positive integer m_i is called the multiplicity of g_i), then we use the polynomial notation $m_1 U^{g_1}+\cdots+m_\ell U^{g_\ell}$ where U is a formal symbol. For example, the multiset value of the invariant for a trefoil with the Alexander quandle $X=\mathbb{Z}_2[t,t^{-1}]/(t^2+t+1)$ with the same coefficient $A=X$ and a certain 2-cocycle is $\{\sqcup_4(1),\sqcup_{12}(t+1)\}$, and is denoted by $4+12U^{(t+1)}$, where we use the convention $U^0=1$ and exponential rules apply.

It is seen that $c=\sum_\tau \epsilon(\tau)(x_\tau,y_\tau)$ is a 2-cycle [8], and the contribution $\sum_\tau \epsilon(\tau)\phi(x_\tau,y_\tau)$ is regarded as the evaluation $\phi(c)$ of the cycle by the cocycle ϕ. Hence a colored knot diagram represents a 2-cycle, and the multiset of evaluations over all colorings is the cocycle invariant. The quandle cocycle invariants have also been defined for knotted surfaces in 4-space, in a similar manner, using quandle 3-cocycles and triple points on projections.

The quandle cocycle invariant using region colorings (sometimes called "*shadow*" colorings) and 3-cocycles were considered in [10]. Let $\phi \in Z^3_{\rm Q}(X;A)$ be a 3-cocycle. Then the weight in this case is defined by $B(\mathcal{C},\tau)=\phi(x_\tau,y_\tau,z_\tau)^{\epsilon(\tau)}$ where $\epsilon(\tau)$ is $+1$ or -1, for a positive or a negative crossing respectively. Then the 3-cocycle invariant is defined by $\Phi_\phi(K)=\sum_{\mathcal{C}\in {\rm Col}_{\rm X}({\rm K})}\prod_\tau B(\mathcal{C},\tau)$ as a state-sum, and by $\{\sum_\tau \epsilon(\tau)\phi(x_\tau,y_\tau,z_\tau) \mid \mathcal{C}\in {\rm Col}_{\rm X}({\rm K})\}$ as a multiset.

3 Quandke Cocycle Invariants of Knots with Polynomial Cocycles

In this section we exhibit examples of quandle cocycle invariants that are obtained from polynomial cocycles. The following particular case illustrates our calculations of the invariant using polynomial cocycles, and will be used in the next section.

Proposition 3.1 *Let $X = \mathbb{Z}_p[t, t^{-1}]/(\xi_m(t))$ for a prime p and a positive integer m, and let $f : X \times X \to X$ be defined by $f(x_1, x_2) = (x_1 - x_2)^{a_1} x_2^{a_2}$ for $a_i = p^{m_i}$, $i = 1, 2$, where m_i are non-negative integers. Suppose that $\xi_m(t)$ divides $1 - t^{(a_1+a_2)}$ in $\mathbb{Z}_p[t, t^{-1}]$. Then the cocycle invariant $\Phi_f(T(2, m))$ of the torus knot $T(2, m)$ is given by the multiset*

$$\Phi_f(T(2, m)) = \{\sqcup_{|X|}(t^2 \xi_m'(t))^{a_2} s^{(a_1+a_2)} \mid s \in X\},$$

where $|X|$ denotes the number of elements of X, and ξ_m' is the derivative of ξ_m. If m is a negative integer, then the invariant consists of the negatives of the invariant for $T(2, |m|)$.

Proof. By Lemma 2.1, indeed $f \in Z^2_{\rm Q}(X; A)$. By assumption any top color vector (a, b) extends to a coloring from Example 2.2. Recall also from Example 2.2 that just below the kth crossing we have the kth color vector $(t\xi_{k-1} + \xi_k b, t\xi_k a + \xi_{k+1} b)$, so the contribution to the invariant is computed as

$$\begin{aligned}
&\sum_{k=1}^{m} f(t\xi_{k-1}a + \xi_k b, t\xi_k a + \xi_{k+1} b) \\
&= \sum_{k=1}^{m} (t\xi_{k-1}a + \xi_k b - t\xi_k a - \xi_{k+1} b)^{a_1} (t\xi_k a + \xi_{k+1} b)^{a_2} \\
&= \sum_{k=1}^{m} [(a-b)(\xi_{k+1} - \xi_k)]^{a_1} [b + t\xi_k(a-b)]^{a_2} \\
&= (a-b)^{a_1} b^{a_2} \sum_{k=1}^{m} (-t)^{ka_1} + (a-b)^{a_1+a_2} t^{a_2} \sum_{k=1}^{m} (-t)^{ka_1} \xi_k^{a_2}.
\end{aligned}$$

The first term is written as

$$(a-b)^{a_1} b^{a_2} \left(\sum_{k=1}^{m} (-t)^k \right)^{a_1} = (a-b)^{a_1} b^{a_2} [-t\xi_m]^{a_1}$$

which vanishes in X. Note that by assumption $t^{a_1+a_2} = 1$ in X, so that $t^{a_1} = t^{-a_2}$. Hence the second term is written as

$$(a-b)^{a_1+a_2} t^{a_2} \left(\sum_{k=1}^{m} (-1)^k t^{-k} \xi_k \right)^{a_2}.$$

Thus we compute $S_m = \sum_{k=1}^{m} (-t)^{-k} \xi_k$. By induction we obtain $S_m = \sum_{k=0}^{m-1} (k+1)(-t)^{k-m}$. On the other hand, we compute

$$\begin{aligned}
S_m &= (-t)^{-m} \left(\sum_{k=0}^{m-1} (-t)^k + \sum_{k=0}^{m-1} k(-t)^k \right) \\
&= (-t)^m \left(\xi_m + t \sum_{k=0}^{m-1} (-1)^k k t^{k-1} \right) \quad = \quad (-t)^m (\xi_m + t\xi_m').
\end{aligned}$$

Hence the contribution of the coloring induced by a top color vector (a, b) is

$$(a-b)^{a_1+a_2}(-t)^{a_2(2-m)}(\xi'_m)^{a_2} = (a-b)^{a_1+a_2}(t^2\xi'_m)^{a_2},$$

since $(-t)^m = 1$ in X. Let $s = a - b$. Then we have

$$\{(a-b)^{(a_1+a_2)} \mid (a,b) \in X \times X\} = \{\sqcup_{|X|} s^{(a_1+a_2)} \mid s \in X\}$$

and the result follows.

If m is negative, then all crossings are negative. Then consider the diagram of $T(2, m)$ that is the mirror of the diagram used above of $T(2, |m|)$, with opposite orientation. Then also consider the colors of $T(2, m)$ at the bottom arcs (a, b). Then the contribution from the coloring induced by this bottom color vector coincides with the negative of the original. Hence the invariant $\Phi_f(T(2, m))$ is the multiset that consists of the negative of $\Phi_f(T(2, |m|))$. $\square$

Example 3.2 For $f(x_1, x_2) = (x_1 - x_2)^4 x_2$, the above formula gives the 2-cocycle invariant

$$\Phi_f(T(2,5)) = \{\sqcup_{16} 0, \sqcup_{80}(t+1), \sqcup_{80}(t^3), \sqcup_{80}(t^3+t+1)\}$$

where $m = 5 = 2^2 + 1$ and $X = \mathbb{Z}_2[t, t^{-1}]/(\xi_5(t))$.

For 3-cocycle invariants, similar calculations can be carried out with the 3-cocycle $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$ where $a_i = p^{m_i}$ for $i = 1, 2$, where m_i's are non-negative integers. In particular, let $X = \mathbb{Z}_p[t, t^{-1}]/(\xi_m(t))$ and $(a, b) \in X \times X$ be a top color vector for a coloring of $T(2, m)$ by X as in Proposition 3.1, with the source region color $c \in X$, where m is a positive integer. Then the contribution to the cocycle invariant of this coloring is given by

$$(a-b)^{a_1+a_2} \sum_{k=1}^{m} (\xi_k(t))^{a_1}(-t)^{k a_2}.$$

In particular, if $t^{a_1+a_2} = 1$ in X, then the contribution is given by

$$(a-b)^{a_1+a_2}(-t)^{a_1(1-m)}(\xi'_m(t))^{a_1} = (a-b)^{a_1+a_2}(-t\xi'_m(t))^{a_1}.$$

Thus we obtain

Proposition 3.3 *Let p be a prime and m be a positive integer. Let $X = \mathbb{Z}_p[t, t^{-1}]/(\xi_m(t))$, and $f : X^3 \to X$ be defined by $f(x_1, x_2, x_3) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}$, where $a_i = p^{m_i}$, for non-negative integers m_i, $i = 1, 2$. If $\xi_m(t)$ divides $1 - t^{a_1+a_2}$, then*

$$\Phi_f(T(2,m)) = \{\sqcup_{|X|^2}(-t\xi'_m(t))^{a_1} s^{a_1+a_2} \mid s \in X\}.$$

Example 3.4 Using Proposition 3.3 and the contribution formula preceding it, we obtain the following list of calculations of 3-cocycle invariants for $T(2, m)$ torus knots carried out by a *Maple* program, for cocycles of the form $f(x, y, z) = (x - y)(y - z)^p$.

- $p=2$, $f(x,y,z)=(x-y)(y-z)^2$.
  - $*$ $m=3$: $16+48\ U^{t}$,
  - $*$ $m=5$: 4096,
  - $*$ $m=7$: 262144,
  - $*$ $m=9$: $4194304+12582912\ U^{(t^4+t^7+1)}$,
  - $*$ $m=11$: 1073741824,
  - $*$ $m=13$: 68719476736,
  - $*$ $m=15$: $1099511627776+3298534883328\ U^{(t^{13}+t^{10}+t^7+t^4+t)}$.
- $p=3$, $f(x,y,z)=(x-y)(y-z)^3$.
  - $*$ $m=3$: $243+486\ U^{(2t+2)}$,
  - $*$ $m=5$: 531441,
  - $*$ $m=7$: 387420489,
  - $*$ $m=9$: $94143178827+188286357654\ U^{(2t^7+2t^6+t^4+t^3+2t+2)}$.
- $p=5$, $f(x,y,z)=(x-y)(y-z)^5$.
  - $*$ $m=3$: $625+3750U^{(t+3)}+3750\ (U^{(4t+2)}+U^{(3t+4)}+U^{(2t+1)})$,
  - $*$ $m=5$: $48828125+97656250\ (U^{(4t^3+2t^2+2t+4)}+U^{(t^3+3t^2+3t+1)})$,
  - $*$ $m=7$: 3814697265625.

Similar calculations are made for twist knots using colorings given in Example 2.2, with polynomial cocycles $f(x,y,z)=(x-y)^{a_1}(y-z)^{a_2}$, to obtain the following.

Proposition 3.5 *The 3-cocycle invariant of the twist knot $k(2n)$ with $2n+2$ crossings as depicted in Fig. 4 is given by*

$$\Phi_f(K)=\{\sqcup_{|X|^2}[-nt^{-a_1}+(1+n(1-t))^{a_1+a_2}]s^2 \mid s\in X\}$$

for $X=\mathbb{Z}_p[t,t^{-1}]/(t-n(1-t)^2)$ and $f(x,y,z)=(x-y)^{a_1}(y-z)^{a_2}$.

Example 3.6 The formula in Proposition 3.5 is input in *Maple* to obtain the following results.

- $p=3$:
  - $*$ $n=1$, $a_1=1$, $a_2=3$: $81+324\ (U^{(t+2)}+U^{(1+2t)})$,
  - $*$ $n=1$, $a_1=3$, $a_2=1$: $81+324\ (U^{(2t+2)}+U^{(t+1)})$,
  - $*$ $n=2$, $a_1=1$, $a_2=3$: $243+486\ U^{(t+1)}$,
  - $*$ $n=2$, $a_1=1$, $a_2=3$: $243+486\ U^{(2t+2)}$.
- $p=5$:
  - $*$ $n=1$, $a_1=1$, $a_2=5$: $3125+6250\ (U^{(3t+3)}+U^{(2t+2)})$,
  - $*$ $n=1$, $a_1=5$, $a_2=1$: $3125+6250\ (U^{(3t+3)}+U^{(2t+2)})$,
  - $*$ $n=2$, $a_1=1$, $a_2=5$: 15625,

* $n=2$, $a_1=5$, $a_2=1$: 15625.
* $n=3$, $a_1=1$, $a_2=5$: $625+3750\ (U^{(t)}+U^{(2t)}+U^{(3t)}+U^{(4t)})$,
* $n=3$, $a_1=5$, $a_2=1$: $625+3750\ (U^{(t+1)}+U^{(2t+2)}+U^{(3t+3)}+U^{(4t+4)})$,
* $n=4$, $a_1=1$, $a_2=5$: $625+3750\ (U^{(t+3)}+U^{(2t+1)}+U^{(3t+4)}+U^{(4t+2)})$.

• $p=7$:

* $n=1$, $a_1=1$, $a_2=7$: $2401+19208\ (U^{(t+3)}+U^{(4t+5)}+U^{(2t+6)}+U^{(5t+1)}+U^{(6t+4)}+U^{(3t+2)})$,
* $n=1$, $a_1=7$, $a_2=1$: $2401+19208\ (U^{(t+1)}+U^{(2t+2)}+U^{(3t+3)}+U^{(4t+4)}+U^{(5t+5)}+U^{(6t+6)})$,
* $n=2$, $a_1=1$, $a_2=7$: 117649,
* $n=2$, $a_1=7$, $a_2=1$: 117649,
* $n=3$, $a_1=1$, $a_2=7$: $2401+19208\ (U^{(3t+4)}+U^{(5t+2)}+U^{(6t+1)}+U^{(t+6)}+U^{(2t+5)}+U^{(4t+3)})$,
* $n=3$, $a_1=7$, $a_2=1$: $2401+19208\ (U^{(t+1)}+U^{(2t+2)}+U^{(3t+3)}+U^{(4t+4)}+U^{(5t+5)}+U^{(6t+6)})$,
* $n=4$, $a_1=1$, $a_2=7$: $2401+19208\ (U^{(t+2)}+U^{(2t+4)}+U^{(3t+6)}+U^{(4t+1)}+U^{(5t+3)}+U^{6t+5)})$,
* $n=4$, $a_1=7$, $a_2=1$: $2401+19208\ (U^{(t+1)}+U^{(2t+2)}+U^{(3t+3)}+U^{(4t+4)}+U^{(5t+5)}+U^{(6t+6)})$,
* $n=5$, $a_1=1$, $a_2=7$: $16807+33614\ (U^{(3t+3)}+U^{(5t+5)}+U^{(6t+6)})$,
* $n=5$, $a_1=7$, $a_2=1$: $16807+33614\ (U^{(t+1)}+U^{(2t+2)}+U^{(4t+4)})$,
* $n=6$, $a_1=1$, $a_2=7$: 117649,
* $n=6$, $a_1=7$, $a_2=1$: 117649.

Remark 3.7 Polynomial cocycles are further utilized in [20] for extensive computer calculations using *Maple* and the knot table. The formulas given in Propositions 3.1 and 3.3, however, enable one to compute for a larger quandles and knots.

Polynomial cocycles were also used in [1] to evaluate cocycle invariants for twist spun knots using Alexander quandles, using formulas in [3]. Such computations are useful in detecting non-invertibility of twist-spun knots.

4 Non-triviality of Quandle Homology Groups

In this section we prove the non-triviality of homology groups of some families of Alexander quandles. The method for dimensions 2 and 3 is to show that there is a coloring of knot diagrams that evaluates non-trivially by a polynomial cocycle constructed in Lemma 2.1. This method has been used repeatedly since [7]. For higher dimensions, we use algebraic machineries developed in [18], so that we give statements and proofs separately.

Proposition 4.1 *The following quandle homology groups $H^m_{\rm Q}(X;X)$ are non-trivial* $(\neq 0)$ *:*

(1) $X=\mathbb{Z}_2[t,t^{-1}]/(\xi_{2^n+1}(t))$ *for any positive integer n, and for $m=2,3$.*
(2) $X=\mathbb{Z}_p[t,t^{-1}]/(\xi_{(p^n+1)/2}(t))$ *for any odd prime p and for any positive integer n, and for $m=2,3$.*
(3) $X=\mathbb{Z}_p[t,t^-1]/(t-n(1-t)^2)$*, for $m=3$, and for:*

   (*a*) $p=3$, $n\equiv 1,2 \pmod 3$,
   (*b*) $p=5$, $n\equiv 1,3,$ *and* $4 \pmod 5$,

(*c*) $p = 7$, *all* n *except* $n \equiv 2, 6 \pmod 7$,
(*d*) $p = 11$, *all* n *except* $n \equiv 1, 2, 6$, *and* $9 \pmod{11}$,
(*e*) $p = 13$, *all* n *except* $n \equiv 2, 4, 6, 7$, *and* $12 \pmod{13}$.

Proof. It is sufficient to show that there is a coloring of a knot contributing a non-trivial value to the quandle cocycle invariant [7].

(1) $m = 2, 3$: For $X = \mathbb{Z}_2[t, t^{-1}]/(\xi_{2^n+1}(t))$, let $f(x_1, x_2) = (x_1 - x_2)^{2^n} x_2$ in Lemma 2.1, so that $a_1 = 2^n$ and $a_2 = 1$. Note that $1 - t^{(a_1+a_2)} = (1 - t)\xi_{2^n+1}(t)$. Take $(1, 0) \in X \times X$ as a top color vector, which extends to a coloring of $T(2, m)$, where $m = 2^n + 1$ by Example 2.2. Then by Propositions 3.1, the contribution to the invariant is a multiple by a power of t of $\xi'_m(t)$, which is non-trivial in X, as the degree of $\xi'_m(t)$ is less than that of $\xi_m(t)$. For the 3-cocycle case, choosing $a_1 = 1$ and $a_2 = 2$ gives a contribution $\xi'_m(t)^{a_1} = \xi'_m(t)$ which is non-zero in X and the same argument applies.

(2) In this case take $a_1 = p^n$ and $a_2 = 1$ as before, then $1-t^{(a_1+a_2)} = (1-t^{(p^n+1)/2})(1+t^{(p^n+1)/2})$. If $(p^n + 1)/2$ is odd, then $\xi_{(p^n+1)/2}(t)$ divides $(1 + t^{(p^n+1)/2})$, and if even, it divides $(1 - t^{(p^n+1)/2})$, hence the result follows by the same argument.

(3) From Proposition 3.5, the invariant is non-trivial if $[-nt^{-a_1} + (1+n(1-t))^{a_1+a_2}]$ that appear in the formula is non-zero. The choice $a_1 = 1$ and $a_2 = p$ gives non-zero values for the cases listed in the statement, among primes $2 < p \leq 29$, and for $0 < n < p$. $\square$

To prove non-triviality for higher dimensions we use the following lemma from [18]. For a quandle $(X, *)$ and a positive integer m, we use the notation $*_a(x) = x * a$ and $(*_a)^m(x) = (\cdots(x * a) * a) * \cdots) * a$ where the operation is performed m times for a positive integer m and for $x, a \in X$. For an n-chain $c = (x_1, \ldots, x_n)$, $x_i \in X$, $i = 1, \ldots, n$ for a positive integer n, the notation $*_a(c) = c * a = (x_1 * a, \ldots, x_n * a)$ is used. The map is extended to the chain groups linearly. The map $h_a : C^{\rm Q}_n(X) \to C^{\rm Q}_{n+1}(X)$ was defined in [18] by linearly extending

$$h_a(c) = h_a((x_1, \ldots, x_n)) = (x_1, \ldots, x_n, a) = (c, a).$$

Then $h'_a : C^{\rm Q}_n(X) \to C^{\rm Q}_{n+1}(X)$ is defined by $h'_a = h_a + *_a h_a + \cdots + (*_a)^m h_a$ for $a \in X$.

Let $s = s(y_0, y_1) = \sum_{i=1}^{k-1}(y_{i-1}, y_i)$ for $y_i \in X$, $i = 0, \ldots, k-1$. Define [18] $h_s : C^{\rm Q}_n(X) \to C^{\rm Q}_{n+2}(X)$ by linearly extending

$$h_s(c) = (c, s) = \sum_{i=1}^{k-1}(x_1, \ldots, x_n, y_{i-1}, y_i).$$

Lemma 4.2 ([18]) (i) *Let X be a quandle such that there is an element $a \in X$ satisfying the condition $(*_a)^m(x) = x$ for any $x \in X$, then h'_a is a chain map.*
(ii) *Let X be a quandle such that there is a sequence of elements $(y_0, \ldots, y_k)$ satisfying the condition $y_{i+1} = y_{i-1} * y_i$ for $i = 1, \ldots, k-1$, $y_0 = y_{k-2} * y_{k-1}$ and $y_1 = y_{k-1} * y_0$. Then h_s is a chain map.*

Theorem 4.3 *For any positive integer n, the quandle $X = \mathbb{Z}_2[t, t^{-1}]/(\xi_m(t))$, where $m = 2^n + 1$, has non-trivial 4-dimensional cohomology: $H^4(X; X) \neq 0$.*

Proof. We exhibit a cycle C and a cocycle f such that $f(C) \neq 0$ to prove non-triviality. Although negative signs are irrelevant in $\mathbb{Z}_2$, we often leave them to indicate computational processes below.

We construct a 4-cycle using a 3-cycle and the map defined above. From a shadow coloring of a $(2, m)$-torus knot with the quandle X such that the top color vector is $(0, 1)$ and the left-most region is colored by 0, we have a 3-cycle $C_3 = \sum_{k=0}^{m-1}(0, \xi_k, \xi_{k+1})$, and define $h_0(C_3) = C_4$ and $h_0'(C_3) = C_4'$. By Lemma 4.2, C_4' is a 4-cycle. We use the 4-cocycle

$$f(x_1, x_2, x_3, x_4) = (x_1 - x_2)^{a_1}(x_2 - x_3)^{a_2}(x_3 - x_4)^{a_3}, \quad a_i = 2^{n_i}, \quad i = 1, 2, 3.$$

Using

$$(x_i * b - x_{i+1} * b)^{a_i} = ((tx_i + (1-t)b) - (tx_{i+1} + (1-t)b))^{a_i} = t^{a_i}(x_i - x_{i+1}),$$

we have

$$f(C_4') = (1 + t^{a_1+a_2+a_3} + \cdots + t^{(m-1)(a_1+a_2+a_3)})f(C_4),$$

where $f(C_4) = \sum_{k=0}^{m-1}(0 - \xi_k)^{a_1}(\xi_k - \xi_{k+1})^{a_2}\xi_{k+1}^{a_3}$. Suppose that $1 - t^{a_1+a_2+a_3} = 0$, then

$$\begin{aligned} f(C_4') &= m\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1}(1 - t\xi_k)^{a_3} \\ &= m\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1} - t^{a_3}\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1+a_3}. \end{aligned}$$

Set $a_1 = a_2 = 2^{n-1}$, $a_3 = 1$ and $m = 2^n + 1$, then $1 - t^{a_1+a_2+a_3} = 1 - t^{2^n+1} = 0$. Then

$$f(C_4') = (\sum_{k=0}^{m-1}(-t)^k\xi_k)^{a_1} - t^{a_3}\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1+a_3}.$$

By induction we see $\sum_{k=0}^{m-1}(-t)^k\xi_k = \dfrac{\xi_m\xi_{m+1}}{1-t}$ (shown in [1]). Since m is odd, ξ_{m+1} in the RHS is divisible by $(1-t)$, and this sum is 0 in X. Then in X

$$f(C_4') = -t^{a_3}\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1+a_3}.$$

We now compute the sum $\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1+a_3}$. Note that $1+t$ is invertible in $X = \mathbb{Z}_2[t, t^{-1}]/(\xi_m(t))$, since $1 + t + t^2 + \cdots + t^{m-1} = 0$ implies $(1+t)(t + t^3 + \cdots + t^{m-2}) = 1$. Hence $(1 + t^{a_1}) = (1+t)^{a_1}$ and $(1 + t^{a_3})$ are invertible. Then we compute

$$\begin{aligned} (1 + t^{a_1})(1 + t^{a_3})\sum_{k=0}^{m-1}(-t)^{ka_2}\xi_k^{a_1+a_3} &= \sum_{k=0}^{m-1}(-t)^{ka_2}(1 - (-t)^{ka_1})(1 - (-t)^{ka_3}) \\ &= \sum_{k=0}^{m-1}(-t)^{k(a_1+a_2+a_3)} = m \end{aligned}$$

since $\sum_{k=0}^{m-1}(-t)^{ka_2} = (\xi_m)^{a_2} = 0$, $\sum_{k=0}^{m-1}(-t)^{ka_3} = (\xi_m)^{a_3} = 0$, and $(-t)^{k(a_2+a_3)} = (-t)^{k(-a_1)}$. Thus $f(C_4') = -mt^{a_3}(1 + t^{a_1})^{-1}(1 + t^{a_3})^{-1} \neq 0$ in X. $\square$

Theorem 4.4 *For any positive integer n and $r > 1$, the quandle $X = \mathbb{Z}_p[t, t^{-1}]/(\xi_m(t))$ has non-trivial $(2r+1)$-dimensional cohomology, $H^{2r+1}(X;X) \neq 0$, if:*

(i) *$p = 2$ and $m = 2^n + 1$, or,*

(ii) *p is an odd prime and $m = (p^n + 1)/2$.*

Proof. We construct a $(2r+1)$-cycle using the $(2r-1)$-cycle C_{2r-1} and the chain map h_s defined above, where $s = \sum_{k=0}^{m-1}(\xi_k, \xi_{k+1})$. Let $C_{2r+1} = h_s(C_{2r-1})$. To compute the cocycle values, denote C_{2r-1} by a formal sum $\sum(x_1, \ldots, x_{2r-1})$ and C_{2r+1} by $\sum_{k=0}^{m-1}\sum(x_1, \ldots, x_{2r-1}, \xi_k, \xi_{k+1})$. By Lemma 4.2, C_{2r+1} is a $(2r+1)$-cycle. We use the $(2r+1)$-cocycle

$$f_{2r+1}(x_1, \ldots, x_{2r+1}) = (x_1 - x_2)^{a_1} \cdots (x_{2r} - x_{2r+1})^{a_{2r}},$$

for $k = 1, \ldots, r$. Then one computes

$$\begin{aligned} f(C_{2r+1}) &= \sum_{j=0}^{m-1} \sum f(x_1, \ldots, x_{2r-1}, \xi_j, \xi_{j+1}) \\ &= \sum_{j=0}^{m-1} \sum (x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}} (x_{2r-1} - \xi_j)^{a_{2r-1}} (\xi_j - \xi_{j+1})^{a_{2r}} \\ &= \sum_{j=0}^{m-1} \sum (x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}} (x_{2r-1})^{a_{2r-1}} (\xi_j - \xi_{j+1})^{a_{2r}} \\ &- \sum_{j=0}^{m-1} \sum (x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}} (\xi_j)^{a_{2r-1}} (\xi_j - \xi_{j+1})^{a_{2r}}. \end{aligned}$$

The first term vanishes because $\sum_{j=0}^{m-1}(\xi_j - \xi_{j+1})^{a_{2r}} = \xi_m^{a_{2r}} = 0$ in X. If a_{2r-1} is odd, then the second term is

$$\sum (x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}} \sum_{j=0}^{m-1} (0 - \xi_j)^{a_{2r-1}} (\xi_j - \xi_{j+1})^{a_{2r}}.$$

Now we prove the theorem by induction by proving that $f_{2r+1}(C_{2r+1})$ is invertible in X under the assumption $f_{2r-1}(C_{2r-1})$ is invertible.

For the case (i), let $a_{2k-1} = 1$ and $a_{2k} = 2^n$. For $r = 1$, $f_3(C_3) = \sum_{j=0}^{m-1}(0 - \xi_j)(\xi_j - \xi_{j+1})^{2^n} = t\xi'_m$, and $t(1+t)\xi'_m(t) = 1$ in X. Assume $f_{2r-1}(C_{2r-1}) = \sum_{j=0}^{m-1}\sum(x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}}$ is invertible in X. Then the above second term is $f_{2r-1}(C_{2r-1})f_3(C_3)$, which is invertible in X and this case is proved.

For the case (ii), let $a_{2k-1} = 1$ and $a_{2k} = p^n$. For $r = 1$, $f_3(C_3) = \sum_{j=0}^{m-1}(0 - \xi_j)(\xi_j - \xi_{j+1})^{p^n} = t\xi'_m$. Differentiating both sides of $(1+t)\xi_m = 1 - (-t)^m$, we obtain $(1+t)\xi'_m = -\xi_m - m(-1)^m t^{m-1} = -mt^{-1}$ in X. By the assumption m is invertible in $\mathbb{Z}_p$, hence $t\xi'_m$ is invertible in X. Assume $f_{2r-1}(C_{2r-1}) = \sum_{j=0}^{m-1}\sum(x_1 - x_2)^{a_1} \cdots (x_{2r-2} - x_{2r-1})^{a_{2r-1}}$ is invertible in X. Then the above term $f_{2r-1}(C_{2r-1})f_3(C_3)$ is invertible in X and the theorem is proved. $\square$

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